Solving Optimization Problems

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Introduction to Solving Optimization Problems

Today's Topics

- Concept of optimization problem
- Defining an optimization problem
- Various types of optimization problems
- Traditional approaches to solve optimization problems
- Limitations of the traditional approaches

Concept of optimization problem

- Optimization : Optimum value that is either minimum or maximum value.

$$y = F(x)$$

Example:

$$2x - 6y = 11$$

or

$$y = (2x - 11) \div 6$$

Can we determine an optimum value for y?
Similarly, in the following case

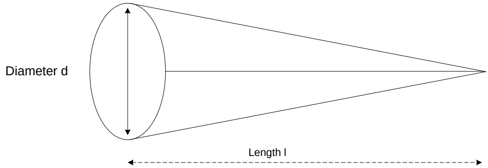
$$3x + 4y \geq 56.$$

These are really not related to optimization problem!

Defining an optimization problem

Suppose, we are to design an optimal pointer made of some material with density ρ. The pointer should be as low weight as possible, with a desirable strength (i.e. sustainable to mechanical breakage) and the deflection of pointing at end should be negligible.

The task is to select the best pointer out of all possible pointers.



Suppose, s is the strength of the pointer.

- Mass of the stick is denoted by
$$M = \frac{1}{3}\Pi\left(\frac{d}{2}\right)^2 * l * \rho = \frac{1}{12}\Pi * d^2 * l * \rho$$
- Deflection : $\delta = f_1(d, l, \rho)$
- Strength : $s = f_2(d, l, \rho)$

Defining an optimization problem

The problem can be stated as

- Objective function

$$\text{Minimize } M = \frac{1}{12}\Pi * d^2 * l * \rho$$

- Subject to

$$\delta \leq \delta_{th}, \text{ where } \delta_{th} = \text{allowable deflection}$$
$$s \geq s_{th}, \text{ where } s_{th} = \text{required strength}$$

and

$$d_{min} \leq d \leq d_{max}$$
$$l_{min} \leq l \leq l_{max}$$

Defining Optimization Problem

An optimization problem can be formally defined as follows:

- Maximize (or Minimize)

$$y_i = f_i(x_1, x_2, \cdots, x_n)$$
$$\text{where } i = 1, 2 \cdots k,\ k \geq 1$$

- Subject to

$$g_i(x_1, x_2, \cdots x_n)\ ROP_i\ c_i$$

  where $i = 1, 2, ..., j,\ j \geq 0$. ROP_i denotes some relational operator and $c_i = 1, 2, \cdots j$ are some constants.
  and
  x_i ROP d_i, for all i=1,2...n ($n \geq 1$)
  Here, x_i denotes a design parameter and d_i is some constant.

Some Benchmark Optimization Problems

Exercises: Mathematically define the following optimization problems.

- Traveling Salesman Problem
- Knapsack Problem
- Graph Coloring Problem
- Job Machine Assignment Problem
- Coin Change Problem
- Binary search tree construction problem

Types of Optimization Problem

Unconstrained optimization problem
Problem is without any functional constraint.

Example:

Minimize $y = f(x_1, x_2) = (x_1 - 5)^2 + (x_2 - 3)^3$
where $x_1, x_2 \geq 0$

Note: Here, $g_j = NULL$

Types of Optimization Problem

Constrained optimization problem

Optimization problem with at one or more functional constraint(s).

Example:

Maximize $y = f(x_1, x_2, \cdots, x_n)$

Subject to

$g_i(x_1, x_2, \cdots, x_{n_i}) \geq c_i$

where $i = 1, 2, \cdots, k$ and $k > 0$

and

$x_1, x_2, \cdots, x_n$ are design parameters.

Types of Optimization Problem

Integer Programming problem

If all the design variables take some integer values.

Example:

Minimize $y = f(x_1, x_2) = 2x_1 + x_2$

Subject to

$x_1 + x_2 \leq 3$

$5x_1 + 2x_2 \leq 9$

and

x_1, x_2 are integer variables.

Types of Optimization Problem

Real-valued problem
If all the design variables are bound to take real values.

Mixed-integer programming problem
Some of the design variables are integers and the rest of the variables take real values.

Types of Optimization Problem

Linear optimization problem

Both objective functions as well as all constraints are found to be some linear functions of design variables.

Example:

Maximize $y = f(x_1, x_2) = 2x_1 + x_2$

Subject to
$x_1 + x_2 \leq 3$
$5x_1 + 2x_2 \leq 10$
and
$x_1, x_2 \geq 0$

Types of Optimization Problem

Non-linear optimization problem

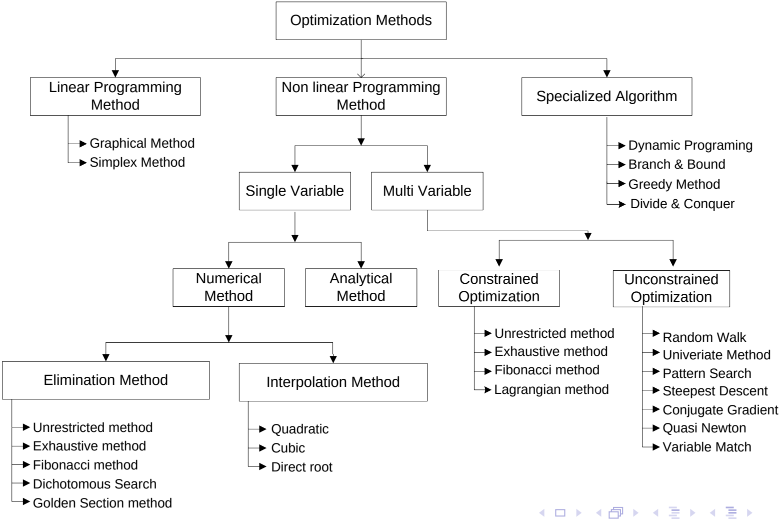
If either the objective function or any one of the functional constraints are non-linear function of design variables.

Example:

Maximize $y = f(x_1, x_2) = x_1^2 + 5x_2^3$

Subject to
$x_1^4 + 3x_2^2 \leq 629$
$2x_1^3 + 4x_2^3 \leq 133$
and
$x_1, x_2 \geq 0$

Traditional approaches to solve optimization problems



Example : Analytical Method

Suppose, the objective function: $y = f(x)$. Let $f(x)$ be a polynomial of degree m and $(m > 0)$

If $y' = f'(x) = 0$ for some $x = x^*$, then we say that
y is optimum (i.e. either minimum or maximum point exist) at the point $x = x^*$.

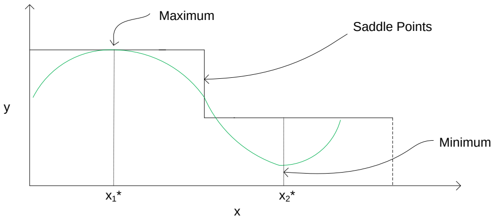
If $y' = f'(x) \neq 0$ for some $x = x^*$, then we say that
there is no optimum value at $x = x^*$ (i.e. $x = x^*$ is an inflection point)

An inflection point is also called a **saddle point**.

Example : Analytical Method

Note:
An inflection point is a point, that is, neither a maximum nor a minimum at that point.

Following figure explains the concepts of minimum, maximum and saddle point.



Example : Analytical Method

Let us generalize the concept of "Analytical method".

If $y = f(x)$ is a polynomial of degree m, then there are m number of candidate points to be checked for optimum or saddle points.

Suppose, y^n is the n_{th} derivative of y.
To further investigate the nature of the point, we determine (first non-zero) $(n - th)$ higher order derivative
$y^n = f^n(x = x^*)$

There are two cases.
Case 1:
If $y^n \neq 0$ for n=odd number, then x^* is an inflection point.

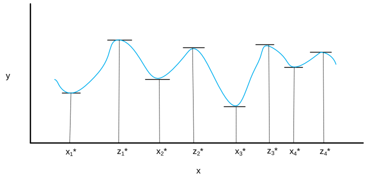
Case 2:
If $y^n = 0$ for $n =$ odd number, then there exist an optimum point at x^*.

Example : Analytical Method

In order to decide the point x^* as minimum or maximum, we have to find the next higher order derivative, that is $y^{n+1} = f^{n+1}(x = x^*)$.
There are two sub cases may be:

Case 2.1:
If $y^{n+1} = f^{n+1}(x = x^*)$ is positive then x is a local minimum point.

Case 2.2:
If $y^{n+1} = f^{n+1}(x = x^*)$ is negative then x is a local maximum point.



If $y^{n+1} = f^{n+1}(x = x^*) = 0$, then we are to repeat the next higher order derivative.

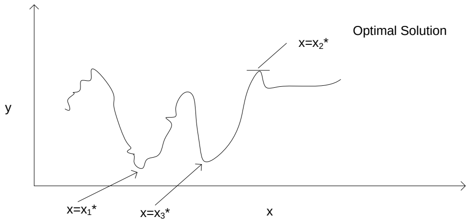
Question

$y = f(x)$

$\frac{d^2y}{dx} = +ve \Rightarrow x = x_1^*$

$\frac{d^4y}{dx} = -ve \Rightarrow x = x_2^*$

$\frac{d^6y}{dx} = +ve \Rightarrow x = x_3^*$



Is the analytical method solves optimization problem with multiple input variables?

If "Yes", than how?

If "No", than why not?

Exercise

Determine the minimum or maximum or saddle points, if any for the following single variable function $f(x)$

$f(x) = \frac{x^2}{2} + \frac{125}{x}$
for some real values of x.

Duality Principle

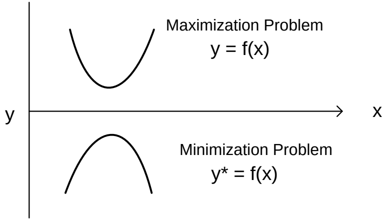
Principle

A minimization (maximization) problem is said to have dual problem if it is converted to the maximization (minimization) problem.

The usual conversion from maximization $\Leftrightarrow$ minimization

$y = f(x) \Leftrightarrow y^* = -f(x)$

$y = f(x) \Leftrightarrow y^* = \frac{1}{f(x)}$



Limitations of the traditional optimization approach

- Computationally expensive.
- For a discontinuous objective function, methods may fail.
- Method may not be suitable for parallel computing.
- Discrete (integer) variables are difficult to handle.
- Methods may not necessarily adaptive.

Soft Computing techniques have been evolved to address the above mentioned limitations of solving optimization problem with traditional approaches.